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MAGNUM P.I.

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ABSTRACT

In an earlier paper Berele and Regev associated to each p.i. algebra A a sequence of algebras $U^{k,l}(A)$ which proved useful in studying the identities of A. We now describe $U^{k,l}(A)$ as a universal object and describe how to recover A from the $U^{k,l}(A)$.

Throughout this paper F will be a field of characteristic zero and all algebras will be algebras over F.

In studying the cocharacters of p.i. algebras in [3] Berele and Regev introduced a construction $U^{k,l}$ which generalized the construction of universal p.i. algebras. If A is a p.i. algebra and $k, l \in \mathbf{N}$, we remind the reader of the construction of $U^{k,l}(A)$ which we now name the magnums of A. First, let $V = T \oplus U$ be a vector space with $\dim T = k$ and $\dim U = l$. The free algebra $F\langle x_1, \ldots, x_k, y_1, \ldots, y_l\rangle$ is identified in a natural way with the tensor algebra of V, which is graded as $\Sigma \oplus W_n$, $W_n = (T \oplus U)^{\otimes n}$. As in [4], W_n is a module for FS_n under the $*$-action. FS_n is identified with the space of multilinear, homogeneous polynomials of degree n in $x_1, \ldots, x_n$ in the usual way, and so defines $I_n(A)$ as the identities of A in FS_n. The subspace $\Sigma \oplus W_n * I_n(A)$ turns out to be an ideal in $F\langle x_1, \ldots, x_k, y_1, \ldots, y_l\rangle$ and $U^{k,l}(A)$ is, by definition, the quotient algebra.

This construction is somewhat indirect, and in this paper we describe $U^{k,l}(A)$ more directly as a certain universal object of A (Theorem 4). We also describe (Theorem 7) how to recover the identities of A from its magnum: we cannot hope to recover A, since the construction of $U^{k,l}$ depends only on the identities of A. As a corollary to Theorem 5, we prove a theorem, also due to Kemer, that for an arbitrary p.i. algebra A, A satisfies all identities of $\mathcal{M}_m(E)$, for large m.

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§1. Graded identities

DEFINITIONS. Let $F\langle X, Y\rangle$ be the free $\mathbf{Z}/2\mathbf{Z}$-graded algebra generated by the set $X \cup Y$, in which elements of X have degree 0 and elements of Y have degree 1. If $A = A_0 \oplus A_1$ is any $\mathbf{Z}/2\mathbf{Z}$-graded algebra and $f(x_1, \ldots, x_k, y_1, \ldots, y_l) \in F\langle X, Y\rangle$, we say that f is a *graded identity* for A if f vanishes under every degree zero homomorphism $F\langle X, Y\rangle \to A$, i.e., if $f(a_1, \ldots, a_k, b_1, \ldots, b_l) = 0$ for all $a_1, \ldots, a_k \in A_0, b_1, \ldots, b_l \in A_1$. For a fixed A, the set Q of graded identities for A in $F\langle X, Y\rangle$ is a *graded T-ideal*, in the sense that Q is invariant under all degree zero homomorphisms $F\langle X, Y\rangle \to F\langle X, Y\rangle$. Note that $\{f(x_1, \ldots, x_n) \in Q \mid x_i \in X, i = 1, \ldots, n\}$ is precisely the set of (ungraded) polynomial identities for A_0.

EXAMPLE. Let E be the infinite dimensional Grassman algebra generated by $e_1, e_2, \ldots$. E has a $\mathbf{Z}/2\mathbf{Z}$-grading, $E = E_0 \oplus E_1$, gotten by setting $e_i \in E_1$ for all i. Then E satisties the graded identities $[x_i, x_j] = [x_i, y_j] = y_iy_j + y_jy_i = 0$.

For the remainder of this paper A will be an ungraded p.i. algebra; $A \otimes E$ will be given the $\mathbf{Z}/2\mathbf{Z}$-grading induced from E, $(A \otimes E)_i = A \otimes E_i$, $i = 0, 1$; and we will be concerned with the graded identities for $A \otimes E$. Let $W = F\langle x_1, \ldots, x_k, y_1, \ldots, y_l\rangle$ be a free $\mathbf{Z}/2\mathbf{Z}$-graded algebra as above. W is $\mathbf{N}$-graded by total degree $W = \sum_{n=0}^{\infty} \oplus W_n$. Each graded component W_n can be identified with $V^{\otimes n}$ as in [4], and so $\mathrm{pl}(k, l)$ acts on W_n as "superderivations."

LEMMA 1. *Let Q = the graded identities of $A \otimes E$. Then $Q \cap W_n$ is a $\mathrm{pl}(k, l)$ submodule of W_n, i.e., if $M \in \mathrm{pl}(k, l)$ then $\tilde{M}(Q) \subseteq Q$.*

PROOF. By linearity it is enough to show that $Q \cap W_n$ is closed under the action of each elementary matrix E_{ij} in $\mathrm{pl}(k, l)$. So, first let M be a degree 0 matrix. For the sake of concreteness, say M is given by $M(x_1) = x_i$, and let $f(x_1, \ldots, x_k, y_1, \ldots, y_l) \in Q \cap W_n$. For every $\alpha \in F$,

$$f(x_1 + \alpha x_i, x_2, \ldots, x_k, y_1, \ldots, y_l) \in Q \cap W_n.$$

By a van der Monde argument the α coefficient of this polynomial will itself be a graded identity, and it equals $\tilde{M}(f)$. Next let M be a degree 1 elementary matrix, and for concreteness take $M(x_1) = y_i$. Now let $0 \neq \alpha \in E_1$, $\alpha^2 = 0$, and consider

$$\begin{aligned} g &= f(x_1 + \alpha y_i, x_2, \ldots, x_k, y_1, \ldots, y_l) \\ &= f(x_1, x_2, \ldots, x_k, y_1, \ldots, y_l) + \alpha f_1(x_1, \ldots, x_k, y_1, \ldots, y_l). \end{aligned}$$

This f_1 vanishes under every graded substitution from $A \otimes E$, hence $f_1 \in Q$. But, since α commutes with each x_i and anticommutes with each y_i, $f_1 = \tilde{M}(f)$, and the lemma follows.

We now need to generalize 2.2–2.4 of [2] and describe how to linearize graded identities. As in [4] W_n is a bimodule for $\mathrm{pl}(k, l)$ and FS_n with the $*$-action, and we denote by B or $B(k, l; n)$ the algebra of endomorphisms of W_n generated by $\mathrm{pl}(k, l)$. If $\omega \in W_n$ is any monomial in $\{x_1, \ldots, x_k, y_1, \ldots, y_l\}$, let

$$R^+ = \{\sigma \in S_n \mid \omega * \sigma = \omega\},$$

$$R^- = \{\sigma \in S_n \mid \omega * \sigma = -\omega\},$$

$$R = R^+ \cup R^- \qquad \text{and}$$

$$s = s(\omega) = \sum_{\sigma \in R^+} \sigma - \sum_{\sigma \in R^-} \sigma \in FS_n.$$

LEMMA 2. *If $\omega = z_1 \cdots z_n \in W_n$ is a monomial and $s(\omega)$, B are as above, then $B\omega = W * s$.*

PROOF. Clearly, $\omega * s = |R| \cdot \omega$, hence

$$B\omega = B\omega * s \subseteq W * s.$$

For the reverse inclusion, since W_n is completely reducible as a B module we decompose W_n over B as

$$W_n = (W_n * s) \oplus W^{(1)} = (B\omega) \oplus W^{(2)} \oplus W^{(1)}$$

Let $\pi : W_n \to W^{(2)}$ be the projection map. We need to show that $\pi = 0$. Since π is a B-map, the double centralizer theorem (4.15 in [4]) implies that there is an $a \in FS_n$ such that $\pi(v) = v * a$ for all $v \in W_n$. Write $a = \sum_{\sigma \in S_n} \alpha_\sigma \sigma$, let T be a transversal for R in S_n and let $J = \{i \mid z_i \text{ has degree } 1\}$, so that $\omega * \sigma = f_J(\sigma)\omega\sigma$ (permutation action) for all $\sigma \in S_n$. Now calculate

$$\begin{aligned} 0 = \pi(\omega) = \omega * a &= \sum_{\sigma \in S_n} \alpha_\sigma \omega * \sigma \\ &= \sum_{\tau \in T} \sum_{\rho \in R} \alpha_{\rho\tau} (\omega * \rho) * \tau \\ &= \sum_{\tau \in T} \sum_{\rho \in R} \alpha_{\rho\tau} f_J(\rho) \omega * \tau. \end{aligned}$$

Since the $\{\omega * \tau\}$ are linearly independent,

$$\sum_{\rho \in R} \alpha_{\rho\tau} f_J(\rho) = 0 \qquad \text{for each } \tau \in T.$$

Finally

$$\begin{aligned}\pi(v*s) &= v*sa\\ &= v*\sum_{\tau\in T}\sum_{\rho\in R}\alpha_{\rho\tau}s\rho\tau\\ &= v*\sum_{\tau\in T}\Big(\sum_{\rho\in R}\alpha_{\rho\tau}f_J(\rho)\Big)s\tau = 0.\end{aligned}$$

Hence $\pi = 0$, proving the lemma.

NOTATION. We make the usual identification $\mathrm{FS}_n \equiv V_n$ = the space of (degree zero) multilinear, homogeneous degree n polynomials in $x_1,\dots,x_n$.

COROLLARY 3. *Let $\omega \in W_n$ be a monomial in $\{x_1,\dots,x_k,y_1,\dots,y_l\}$, $a \in V_n \equiv \mathrm{FS}_n$ and $s = s(\omega)$ as above. Then $\omega * a$ is a graded identity for $A \otimes E$ if and only if sa is a polynomial identity for A.*

PROOF. First, assume that $\omega * a \in Q$, the graded identities for $A \otimes E$. Lemma 2 implies that for some $b \in B(k,l;n)$ (increasing k if necessary) $b\omega = x_1x_2\cdots x_n * s$, hence, by Lemma 1, $x_1\cdots x_n * sa \in Q$. But $x_1\cdots x_n \in V_n$ and, in the identification of V_n with FS_n, $x_1\cdots x_n$ corresponds to the identity, hence $sa \in Q \cap V_n$.

To prove the reverse inclusion, note that Lemma 2 implies that $B(x_1x_2\cdots x_n) = W$ (again, increasing k if necessary), hence $b(x_1\cdots x_n) = \omega$ for some $b \in B$. Now if $sa \in Q \cap V_n$, then as above $(x_1\cdots x_n)*sa \in Q$ so

$$\begin{aligned}\omega * a &= \frac{1}{|R|}\omega * sa\\ &= \frac{1}{|R|}b(x_1\cdots x_n)*sa \in Q.\end{aligned}$$

We now prove the main result of this section.

THEOREM 4. *For any algebra A and $k,l \in \mathbf{N}$, the magnum $U^{k,l}(A) = F\langle x_1,\dots,x_k,y_1,\dots,y_l\rangle$ modulo the graded identities for $A \otimes E$.*

PROOF. Let Q = graded identities of A and $I_n = Q \cap V_n$. Let $a \in I_n$, $\omega \in W_n$ a monomial and $s = s(\omega)$ as above. Then $sa \in I_n$ since I_n is a right ideal of FS_n and so $\omega * a \in Q$ by Lemma 3.

Conversely, let $v \in W$ be a graded identity for $A \otimes E$. By a van der Monde argument, we may assume that $v = \omega * a$ for ω a monomial, i.e., we may assume that v is homogeneous. Again taking $s = s(\omega)$, $\omega * s = |R|\,\omega$ and so

$$\omega * a = \frac{1}{|R|}\omega * sa.$$

Finally, by Lemma 3, $sa \in I_n$ and therefore $\omega * a \in W_n * I_n$. So $\Sigma \bigoplus W_n * I_n = Q \cap W$ and the theorem follows.

§2. Magnum

Theorem 4 describes the magnum of A in terms of A. In this section we construct from the magnum of A an algebra which satisfies the same identities as A. We need some preliminaries.

DEFINITION. $H(k, l; n)$ denotes the set of partitions

$$\{\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathrm{Par}(n) \mid \lambda_{k+1}, \lambda_{k+2}, \ldots \leqq l\}.$$

The group ring FS_n decomposes into a direct sum of two sided ideals indexed by $\mathrm{Par}(n)$,

$$FS_n = \sum_{\lambda \in \mathrm{Par}(n)} \bigoplus I_\lambda.$$

and we break this up as

$$FS_n = \Big(\sum_{\lambda \in H(k,l;n)} \bigoplus I_\lambda\Big) \bigoplus \Big(\sum_{\lambda \notin H((k,l;n)} \bigoplus I_\lambda\Big) =_{\mathrm{DEF}} C(k, l; n) \bigoplus D(k, l; n).$$

We will need

THEOREM 5 (Amitsur–Regev [1]). *Let A be any p.i. algebra. Then there exists $k, l \in \mathbf{N}$ depending on A, such that for all n and all $a \in D(k, l; n)$, a is an identity for A.*

Lemma 6 is analogous to the proof of theorem 14 in [3].

LEMMA 6. *Let $\Sigma_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$ be an identity for A, $\omega_1, \ldots, \omega_n$ monomials in $F\langle x_1, \ldots, x_k, y_1, \ldots, y_l\rangle$ and $J = \{i \mid \omega_i$ has $(\mathbf{Z}/2\mathbf{Z})$ degree $1\}$. Then $\Sigma_{\sigma \in S_n} \alpha_\sigma f_J(\sigma)\omega_{\sigma(1)} \cdots \omega_{\sigma(n)}$ is a graded identity for $A \otimes E$.*

PROOF. Let d = degree (in $\mathbf{N}$) of the product $\omega_1 \cdots \omega_n$, and factor $x_1 \cdots x_d$ as $v_1 \cdots v_n$ so that $\deg v_i = \deg \omega_i$, $i = 1, \ldots, n$. Then $\Sigma \alpha_\sigma v_{\sigma(1)} \cdots v_{\sigma(n)}$ is an identity for A and it is easy to see that

$$\sum \alpha_\sigma f_J(\sigma)\omega_{\sigma(1)} \cdots \omega_{\sigma(n)} = (\omega_1 \cdots \omega_n) * \sum \alpha_\sigma v_{\sigma(1)} \cdots v_{\sigma(n)} \in W_d * I_d(A).$$

By Theorem 4 this is a graded identity for $A \otimes E$.

The $\mathbf{Z}/2\mathbf{Z}$-grading of W induces a $\mathbf{Z}/2\mathbf{Z}$-grading on the magnum $U^{k,l}(A)$. Hence $U^{k,l}(A)\otimes E$ may be regarded either as a $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$-graded algebra or as a $\mathbf{Z}/2\mathbf{Z}$-graded algebra.

THEOREM 7. (cf. [7] prop. 2) *Let A be any p.i. algebra and let k, l be as in Theorem 5. Then A satisfies the same set of identities as*

$$(U^{k,l}(A)\otimes E)_0=(U^{k,l}(A)_0\otimes E_0)\otimes(U^{k,l}(A)_1\otimes E_1).$$

PROOF. Denote $U=(U^{k,l}(A)\otimes E)_0$, $I(A)=$ the identities of A and $I(U)=$ the identities of U. To prove the theorem it is enough to prove that $I(U)\cap V_n=I(A)\cap V_n$ for all n.

First, let $g=\Sigma\,\alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}\in I(A)$. By linearity of g, we need to prove that g vanishes under substitutions of the form $x_i\to\omega_i\otimes\varepsilon_i\in U$, $\omega_i\in U^{k,l}(A)$ is a monomial, $\varepsilon_i\in E$. Fix such a substitution and let

$$J=\{i\mid\omega_i\otimes\varepsilon_i\in U^{k,l}(A)_1\otimes E_1\}.$$

Now calculate

$$\begin{aligned}g(\omega_1\otimes\varepsilon_1,\dots,\omega_n\otimes\varepsilon_n)&=\sum\alpha_\sigma\omega_{\sigma(1)}\cdots\omega_{\sigma(n)}\otimes\varepsilon_{\sigma(1)}\cdots\varepsilon_{\sigma(n)}\\&=\sum(\alpha_\sigma f_J(\sigma)\omega_{\sigma(1)}\cdots\omega_{\sigma(n)})\otimes\varepsilon_1\cdots\varepsilon_n\\&=0\end{aligned}$$

by Lemma 6.

Conversely, let $g\in I(U)\cap V_n=I_n(U)$. Decompose V_n over FS_n as $V_n=I_n(A)\oplus K_n$ and assume by way of contradiction that $g\in K_n$. By [3], $W_n=W_n*I_n(A)\oplus W_n*K_n$. Moreover, by Theorem 5, $g\in C(k,l;n)$ and so by the Hook Theorem (3.18, [4]) $\omega*g\neq0$ for some monomial $\omega\in W_n$. In particular $\omega*g\notin W_n*I_n(A)$.

On the other hand, for this particular $\omega=z_1\cdots z_n$ choose $\varepsilon_1,\dots,\varepsilon_n\in E$ such that $\varepsilon_1\cdots\varepsilon_n\neq0$ and each $\deg\varepsilon_i=\deg z_i$ it the $\mathbf{Z}/2\mathbf{Z}$-grading. Then each $z_i\otimes\varepsilon_i\in U$ and so $g(z_1\otimes\varepsilon_1,\dots,z_n\otimes\varepsilon_n)=0$. Write g as $\Sigma_{\sigma\in S_n}\alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}$. If $J=\{i\mid\deg\varepsilon_i=1\}$, then, as before,

$$\left(\sum\alpha_\sigma f_J(\sigma)z_{\sigma(1)}\cdots z_{\sigma(n)}\right)\otimes\varepsilon_1\cdots\varepsilon_n=0$$

and so $\Sigma\,\alpha_\sigma f_J(\sigma)z_{\sigma(1)}\cdots z_{\sigma(n)}=0$ in $U^{k,l}(A)$. But this latter equals $\omega*g$, giving the contradiction $\omega*g\in W_n*I_n(A)$.

REMARK. $U^{k,l}(A)$ is a finitely generated algebra, and so Theorem 7 relates the identities of arbitrary p.i. algebras to identities of finitely generated algebras. As one corollary we get the following theorem of Kemer [5].

COROLLARY 8. *If A is any p.i. algebra, then for some $m>0$, $I(A)\supseteq I(\mathcal{M}_m(E))$.*

PROOF. By Theorem 7,

$$I(A)=I(U)\supseteq I(U^{k,l}(A)\otimes E).$$

$U^{k,l}(A)$ is a finitely generated algebra, hence for large m, $I(U^{k,l}(A))\supseteq I(\mathcal{M}_m(F))$ ([6]) and so $I(A)\supseteq I(\mathcal{M}_m(F)\otimes E)=I(\mathcal{M}_m(E))$.

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